Problem Set 3
Optical Waveguides and Fibers (OWF)

will be discussed in the tutorial on November 17, 2015

Exercise 1: Transfer matrix formalism

Consider a stack of alternating layers consisting of five layers of silicon dioxide ($n_{\mathrm{SiO}_2} = 1.44$ at $\lambda \approx 1.55$µm) and four layers of silicon ($n_{\mathrm{Si}} = 3.48$ at $\lambda \approx 1.55$µm), as shown in Fig. 1. Let the thicknesses of the layers be $d_{\mathrm{SiO}_2} = 796\,\mathrm{nm}$ and $d_{\mathrm{Si}} = 330\,\mathrm{nm}$. The layer stack is embedded in air ($n_{\mathrm{air}} = 1$) that corresponds to regions 1 and 11 in Fig. 1. In this problem set, we are going to calculate the wavelength-dependent power transmission $\tau(\lambda)$ and power reflection $\rho(\lambda)$, when a plane wave impinges orthogonally on one side of the layer stack.

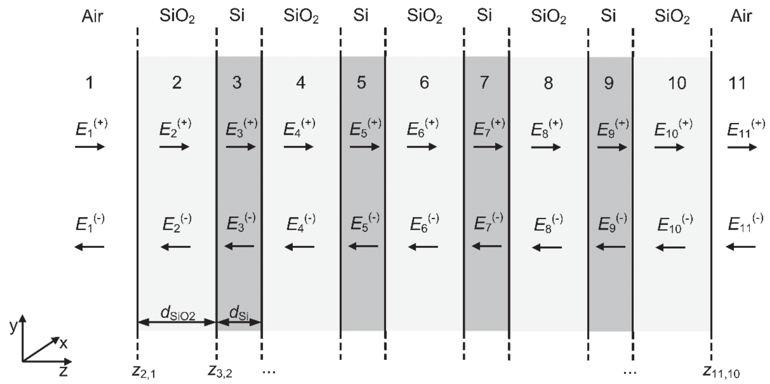

Figure 1: The layer stack and the corresponding coordinate system.

The amplitudes $\hat{E}_\nu^{(\pm)}$ ($\nu = 1 \ldots N$) denote the foward-propagating (+) and backward-propagating (-) amplitudes in the ν-th region. The electric field is then defined by:

$$\begin{aligned} \underline{\mathbf{E}}_\nu(z,t) &\equiv \mathbf{e}_x \left[\hat{E}_\nu^{(+)} e^{\mathrm{j}(\omega t - n_\nu k_0 z)} + \hat{E}_\nu^{(-)} e^{\mathrm{j}(\omega t + n_\nu k_0 z)} \right] \\ &\equiv \mathbf{e}_x \left[E_\nu^{(+)}(z) e^{\mathrm{j}\omega t} + E_\nu^{(-)}(z) e^{\mathrm{j}\omega t} \right] \end{aligned} \tag{1}$$

where k_0 is the propagation constant in vacuum and $\mathbf{e}_x$ is the unit vector along x representing an x-polarized wave without loss of generality.

a) Consider first a *single* planar boundary located at $z_{2,1}$ separating two semi-infinite regions having indices of refraction n_1 and n_2. Assume that two waves having amplitudes $E_1^{(+)}(z)$ and $E_2^{(-)}(z)$ are impinging on the boundary from the left and the right. Show that the amplitudes of the outgoing waves can be expressed in terms of the ingoing amplitudes by:

$$\begin{pmatrix} E_1^{(-)}(z_{2,1}) \\ E_2^{(+)}(z_{2,1}) \end{pmatrix} = \begin{pmatrix} \frac{n_1-n_2}{n_1+n_2} & \frac{2n_2}{n_1+n_2} \\ \frac{2n_1}{n_1+n_2} & \frac{n_2-n_1}{n_1+n_2} \end{pmatrix} \begin{pmatrix} E_1^{(+)}(z_{2,1}) \\ E_2^{(-)}(z_{2,1}) \end{pmatrix} \tag{2}$$

To this end, you can make use of the linearity of Maxwell's equations and the equations you derived in the lecture when a *single* plane wave is impinging on a boundary x.

b) In order to easily describe how the amplitudes propagate through the layer, it is useful to express the amplitudes at the right boundary of the layer in terms of those at the left boundary. Show that the following relations hold:

$$\begin{pmatrix} E_2^{(+)}(z_{2,1}) \\ E_2^{(-)}(z_{2,1}) \end{pmatrix} = \mathbf{T}_{2,1} \begin{pmatrix} E_1^{(+)}(z_{2,1}) \\ E_1^{(-)}(z_{2,1}) \end{pmatrix} \tag{3}$$

$$\text{with} \quad \mathbf{T}_{2,1} = \begin{pmatrix} \frac{n_1+n_2}{2n_2} & \frac{n_2-n_1}{2n_2} \\ \frac{n_2-n_1}{2n_2} & \frac{n_1+n_2}{2n_2} \end{pmatrix} \tag{4}$$

c) Substituting the definitions $E_\nu^{(\pm)}(z) = \hat{E}_\nu^{(\pm)} e^{\mp \mathrm{j} n_\nu k_0 z}$, show that the propagation through two layers and the associated interface can be described by:

$$\begin{pmatrix} \hat{E}_2^{(+)} \\ \hat{E}_2^{(-)} \end{pmatrix} = \begin{pmatrix} e^{\mathrm{j}n_2 k_0 z_{2,1}} & 0 \\ 0 & e^{-\mathrm{j}n_2 k_0 z_{2,1}} \end{pmatrix} \mathbf{T}_{2,1} \begin{pmatrix} e^{-\mathrm{j}n_1 k_0 z_{2,1}} & 0 \\ 0 & e^{\mathrm{j}n_1 k_0 z_{2,1}} \end{pmatrix} \begin{pmatrix} \hat{E}_1^{(+)} \\ \hat{E}_1^{(-)} \end{pmatrix}. \tag{5}$$

d) Finally, generalizing and combining all the previous results, show that one can write for the entire layer stack:

$$\begin{pmatrix} E_{11}^{(+)}(z_{11,10}) \\ E_{11}^{(-)}(z_{11,10}) \end{pmatrix} = \mathbf{M} \begin{pmatrix} E_1^{(+)}(z_{2,1}) \\ E_1^{(-)}(z_{2,1}) \end{pmatrix} \tag{6}$$

where

$$\mathbf{M} = \mathbf{T}_{N,N-1}\, \mathbf{P}_{N-1} \cdots\, \mathbf{P}_2\, \mathbf{T}_{2,1}$$

is the total transmission matrix, and where

$$\mathbf{P}_\nu = \begin{pmatrix} e^{-\mathrm{j}n_\nu k_0 d_\nu} & 0 \\ 0 & e^{\mathrm{j}n_\nu k_0 d_\nu} \end{pmatrix}$$

represents the propagator through the ν-th region that has thickness d_ν.

e) The relation between the fields at the beginning and the end of the layer stack is given by

$$\begin{pmatrix} E_\nu^{(+)}(z_{\nu,\nu-1}) \\ E_\nu^{(-)}(z_{\nu,\nu-1}) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_1^{(+)}(z_{2,1}) \\ E_1^{(-)}(z_{2,1}) \end{pmatrix}.$$

With the help of a numerical language (e.g., MATLAB), plot the power transmission coefficient from right to left $\tau(\lambda) = \frac{|\hat{E}_1^{(-)}|^2}{|\hat{E}_{11}^{(-)}|^2} = 1/|M_{22}|^2$ and the power reflection coefficient $\rho(\lambda) = \frac{|E_{11}^{(+)}|^2}{|E_{11}^{(-)}|^2} = |M_{12}|^2/|M_{22}|^2$ in the range $\lambda = 1000\,\mathrm{nm}$ to $2000\,\mathrm{nm}$. Notice that the latter equations for τ and ρ are obtained by setting $\hat{E}_1^{(+)} = 0$.
Hint: For simplicity, consider a constant refractive index over the whole wavelength range.

f) Finally, set the value of the refractive index of the silicon dioxide in your numerical code to $n_{\mathrm{SiO_2}} = 1$ (this is equivalent to substituting the silicon dioxide with air) and take a large number of layers (for example 30.) Assume new layer thicknesses of $d_{\mathrm{SiO_2}} = 500\,\mathrm{nm}$ and $d_{\mathrm{Si}} = 1\,\mathrm{nm}$ and repeat the transmission and reflection plots in the range $\lambda = 200\,\mathrm{nm}$ to $3000\,\mathrm{nm}$. What are the largest wavelengths at which you observe a reduced transmission? Can you provide an explanation of this phenomenon?

g) What do you observe if you plot the transmission and the reflection coefficients as functions of the wavenumber k_0?

Questions and Comments:

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